U S A Mathematical Talent Search

PROBLEMS

Round 4 - Year 10 - Academic Year 1998-99

1/4/10. Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

2/4/10. For a nonzero integer i, the exponent of 2 in the prime factorization of i is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2 \times 7$. The numbers $3^n - 1$ for $n = 1, 2, 3, \ldots$ are all even, so $ord_2(3^n - 1) > 0$ for $n > 0$.

a) For which positive integers n is $ord_2(3^n - 1) = 1$?

b) For which positive integers n is $ord_2(3^n - 1) = 2$?

c) For which positive integers n is $ord_2(3^n - 1) = 3$?

Prove your answers.

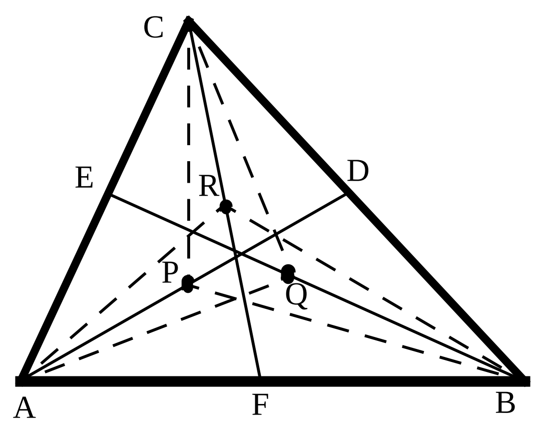
3/4/10. Let f be a polynomial of degree 98, such that $f(k) = \frac{1}{k}$ for $k = 1, 2, 3, \ldots, 99$. Determine $f(100)$.

4/4/10. Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that no two elements of A differ by 6, 9, 12, 15, 18, or 21. Prove that two elements of A must differ by 3.

5/4/10. In ΔABC, let D, E, and F be the midpoints of the sides of the triangle, and let P, Q, and R be the midpoints of the corresponding medians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively, as shown in the figure at the right. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}$$

does not depend on the shape of ΔABC and find that value.



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Complete, well-written solutions to **at least two** of the problems above, accompanied by a completed Cover Sheet, should be sent to the following address and **postmarked no later than March 13, 1999**. Each participant is expected to develop solutions without help from others.

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